



How many ways can you find to solve the problems using the number balance?

What is



Gwen Hancock
Vicky Kukuruda

What is Equal?

Command to carry out a calculation

OR

The relation between two equal quantities



Learning Outcomes:

Engage students in examining their thinking about the meaning of the equal sign

Examine student errors to plan ways to teach the principle of equality

Standards

- **K.OA.1** Represent addition and subtraction with objects, fingers, metal images, drawings, sounds, acting out situations, verbal explanations, expressions, or equations.
- **1.OA.7** Understand the meaning of the equal sign, and determine if equations involving addition and subtraction are true or false.
- **1.OA.8** Determine the unknown whole number in an addition or subtraction equation relating three whole numbers. (ex $8 + ? = 11$)
- **2.OA.1** Use addition and subtraction within 100 to solve one and two-step word problems with unknowns in all positions.



Math Practice Standards

- MP 5 Attend to precision -- State the meaning of symbols including the equal sign consistently and appropriately



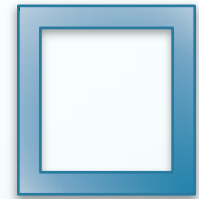
- MP 3 Construct viable arguments and critique the reasoning of others



How would your students respond?

$$8 + 4 = \square + 7$$

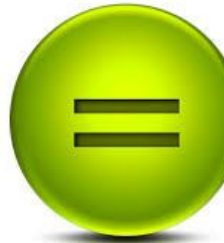
$$5868 + 4 = 5863 +$$



Development of Concepts of Equality

- **Preschoolers** start by counting objects on both sides of the equal sign to compare the numbers in each set
 - Compares objects in two sets without counting
- **Elementary** students interpret $+$ and $=$ in terms of actions to be performed
 - Hinders understanding of $3 = 3$
- **Ages 6-10** need to “see” the relationship between numbers before understanding (ex. $4 + 5 = 6 + 3$)
- **Ages 10 – 13** may not need to make the actual replacement to guarantee understanding
- **Age 13 +** infer beyond the physical models to form generalizations regarding equality

Why Equals Matters



- The equals sign indicates *equality*. It is placed between *two things of equal value*. Think of it as a balance.
- When you say $8 + 3 = 11$, you are saying that $8 + 3$ **has the same value** as 11.
- Knowing this in elementary grades will help students better understand algebraic equations
 - When students are learning about evaluating algebraic equations such as $3x + 7 = 28$ they **MUST** understand that both sides are equal or they will never understand how to solve for x . *The time to teach this is NOT in middle school but in the early grades.*

Why do we care what Equal means?

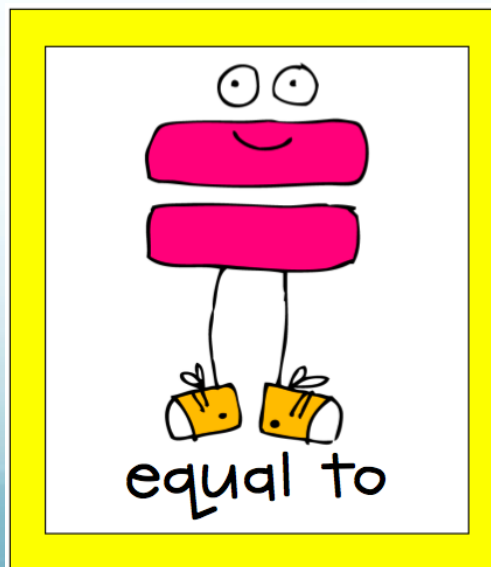


Read the article, *Children's Understanding of Equality: A Foundation for Algebra*

- Share out to answer the question on the slide title
- Discuss the misconceptions mentioned in the article
- Compare to your students in your classroom

What do you notice about the responses to $8 + 4 = \blacksquare + 5$

	Percent Responding			
Grade	7	12	17	12 and 17
1 and 2	5	58	13	8
3 and 4	9	49	25	10
5 and 6	2	76	21	2



Falkner, Levi and Carpenter,
Thinking Mathematically, 1999.

What Do Misconceptions Tell Us?

$$8 + 4 = \boxed{} - 5$$

Juan's Way: $8 + 4 = 12 + 5 = 17$

- “You have to add all the numbers. That’s what it says to do.”
- Juan used the equal sign to show sequences of calculations

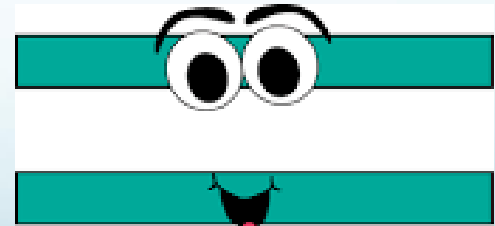
How is that wrong? Remember, both sides *have to be equal*. The *correct way to write it would either include multiple number sentences or parentheses*:

$$(8 + 4) + 5 = 17$$

OR

$$8 + 4 = 12$$

$$12 + 5 = 17$$



Misconceptions



- Calculate answers rather than looking for relationships
 - Generalize that the answer comes after the equal sign
 - Think of it as operation to be carried out rather than way to represent a number
- With small numbers, students often calculate even when they could compare expressions without calculating

What Did You See?



Watch the Video Clip from
Thinking Mathematically.

- What did you see?
- What are the implications for your instructional practice?
- What changes will you consider making in your classroom as a result of what you saw?

What Do We Learn From Children's Reasoning?

$$8 + 4 = \square + 5$$



- Lucy's explanation: I knew that $8 + 4$ is 12 so I had to figure what to go with 5 to make 12.
 - I had to 12 over here so I had to have 12 over there (pointing to the left and the right of the $=$ sign)
- Ricky's explanation: I saw that 5 over here (on the left) was one more than 4 over here (on the right), so the number in the box had to be one less than 8. So it's 7.

MP 5 Express numerical answers with a degree of precision appropriate for the problem context.

Does Understanding the Equal Sign Matter? The Algebra Connection

- Algebra is the gatekeeper to future educational and employment opportunities
- Students have either an “operational” view (total, the answer) or a “relational” view (the same as) of equality
 - Students must understand the equal sign as expressing a relation to make sense of the transformations performed on an equation
 - $5x + 32 = 97$
 - $5x + 32 - 32 = 97 - 32$

$$x + 7 = 10 + 12$$

is the same as

$$x = 10 + 12 - 7$$

Research Study

McNeil and Alibali (2005), found that misconceptions regarding the meaning of the equal sign persist in high school and college students

- Suggests that misconceptions are robust and long lasting

Study involving 177 middle school students focused on:

1. State the meaning of the equal sign

$3 + 4 = 7$, What does the symbol mean?

2. Provide an alternative meaning
Can it mean anything else? Explain.

3. Determine the solution to a typical algebra equation





Study Results



Best Definition	Grade 6	Grade 7	Grade 8
Operational	53	36	52
Relational	32	43	31
Other	15	20	17
No response/ don't know	0	1	0

Students were not more likely to exhibit an understanding of the equal sign as the grade level increased

No significant teacher effects

Students with higher scores on mathematics portion of standardized tests were more likely to have a relational understanding of the equal sign

Percentage of Students Using Various Strategies

Strategy Code	Grade 6	Grade 7	Grade 8
Answer Only	51	25	17
No response/DK	26	24	14
Guess and test	6	15	16
Unwind	9	26	9
Algebra	0	1	33
Other	9	8	12

Guess and Test
(arithmetic)

$$4m + 10 = 70 \quad 20 \times 4 = 80, 15 \times 4 = 60$$

Unwind
(prealgebra)

$$3m + 7 = 25 \quad 25 - 7 = 16, \quad \frac{18}{3} = 6$$

Algebra

$$4m + 10 = 70 \quad 4m + 10 - 10 = 70 - 10$$

$$\frac{4m}{4} = \frac{60}{4} \quad m = 15$$

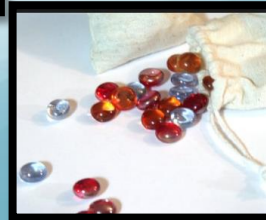
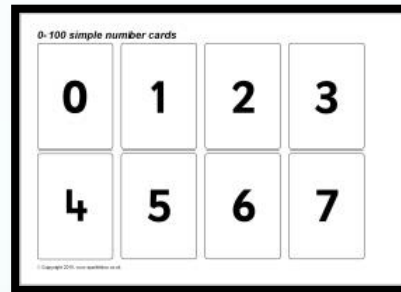
Equation Solving Results

- Students with a relational understanding of equal sign were more likely to solve the equations correctly
 - A relational view of the equal sign is necessary to **interpret** equations and to **operate** on equations
- All students who solved the equations using an algebraic strategy solved correctly
- 8th graders enrolled in algebra were more likely to use an algebraic strategy (86% v. 29%)
- Poor performance on measures of understanding the equal sign indicates lack of explicit focus in elementary and middle school instruction and curricula

Making Equal Expressions

- Make a mat with an equals sign and have students place cards of equal value on either side of the symbol.

- Playing cards
- Number cards
- Picture cards
- Word cards
- Stones
- Bears



Communicating Thinking

Class discussions provide a basis for considering different perspectives on the meaning of equal.

1. Start with a thought-provoking problem
2. Task monitor as students work independently
3. Select students to share work and explanations
4. Discuss

Engage students in productive mathematical arguments

(ex. Be specific about what you think the = sign means)

Students are more capable of reasoning and abstract thinking than we give them credit for

MP 3 Construct viable arguments and critique the reasoning of others



What Can We Do In Elementary Grades?

1. Look for and create opportunities to engage students in conversations about the equal sign
2. Use misconceptions (equality strings) to provide an opportunity to discuss the proper use of the equal sign

$$3 + 5 = 8 + 2 = 10 + 5 = 15$$

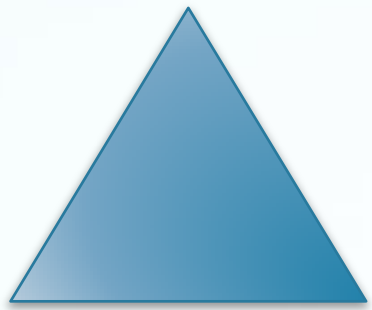
3. Practice problems with numbers and operations on both sides of the equal sign



Where to begin?

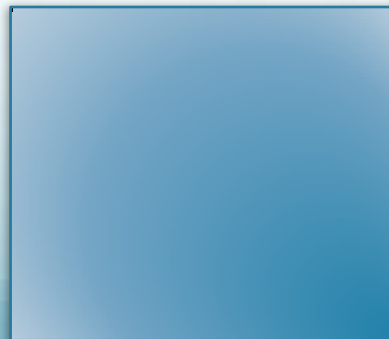
- Use *true and false number sentences* to encourage children to reflect on the meaning of the equals sign
 - Start simple with a single number to the right of =
 - Move to number sentences with the = sign in various positions
 - $3 + 5 = 8$ $3 + 5 = 3 + 5$
 - $8 = 3 + 5$ $3 + 5 = 5 + 3$
 - $8 = 8$ $3 + 5 = 4 + 4$
- Ask children to write their own true and false number sentences
- Use *open number sentences* and vary the location of the unknown variable $\underline{\hspace{1cm}} + 3 = 8$ $8 = 3 + \underline{\hspace{1cm}}$
- Transition to variables such as $a = b + 3$
 - If this sentence is true, which is larger, a or b?





Number Shapes Game

- Each person at the table takes a card and determine what numbers solve the problem on your card
- First person reveals the numbers that correspond to each shape
- The next person plays their card to see if they agree with the numbers chosen by first person.
- Continue play until all numbers on all four agree



Frayer Model

Definition

Having the same amount or value

Characteristics

Attribute that does not change

Relationship between equal quantities

Values on left and right side of = are the same

Equality

Examples

$$3=3$$

$$4 + 5 = 6 + 3$$

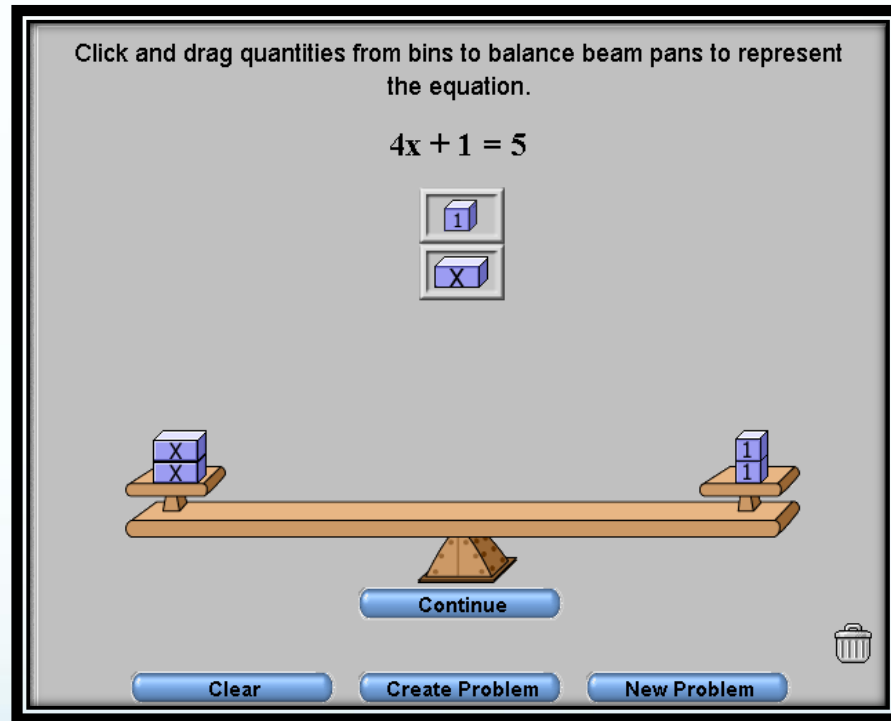
$$201 + 5 = 199 + 7$$

Non-examples

$$6 + 8 = 5 + \square$$

$$6 + 8 + 5 = 19$$

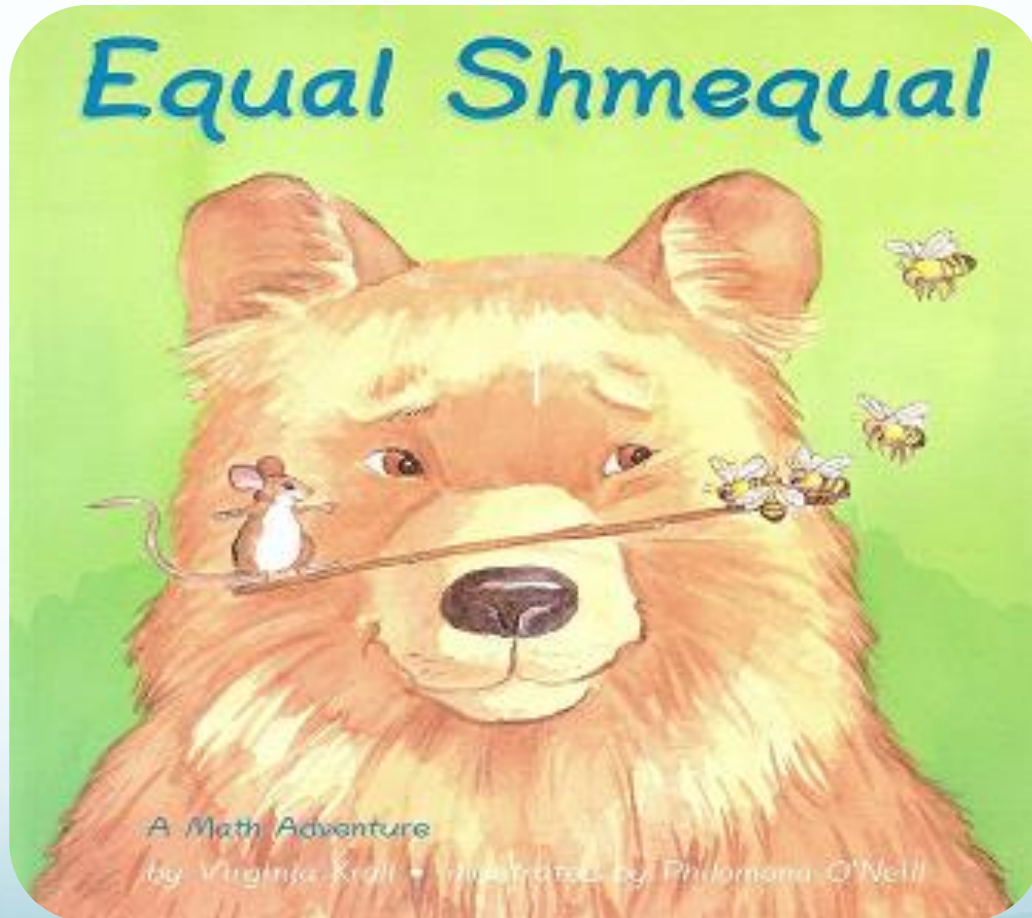
Visual Representation of Algebraic Expressions



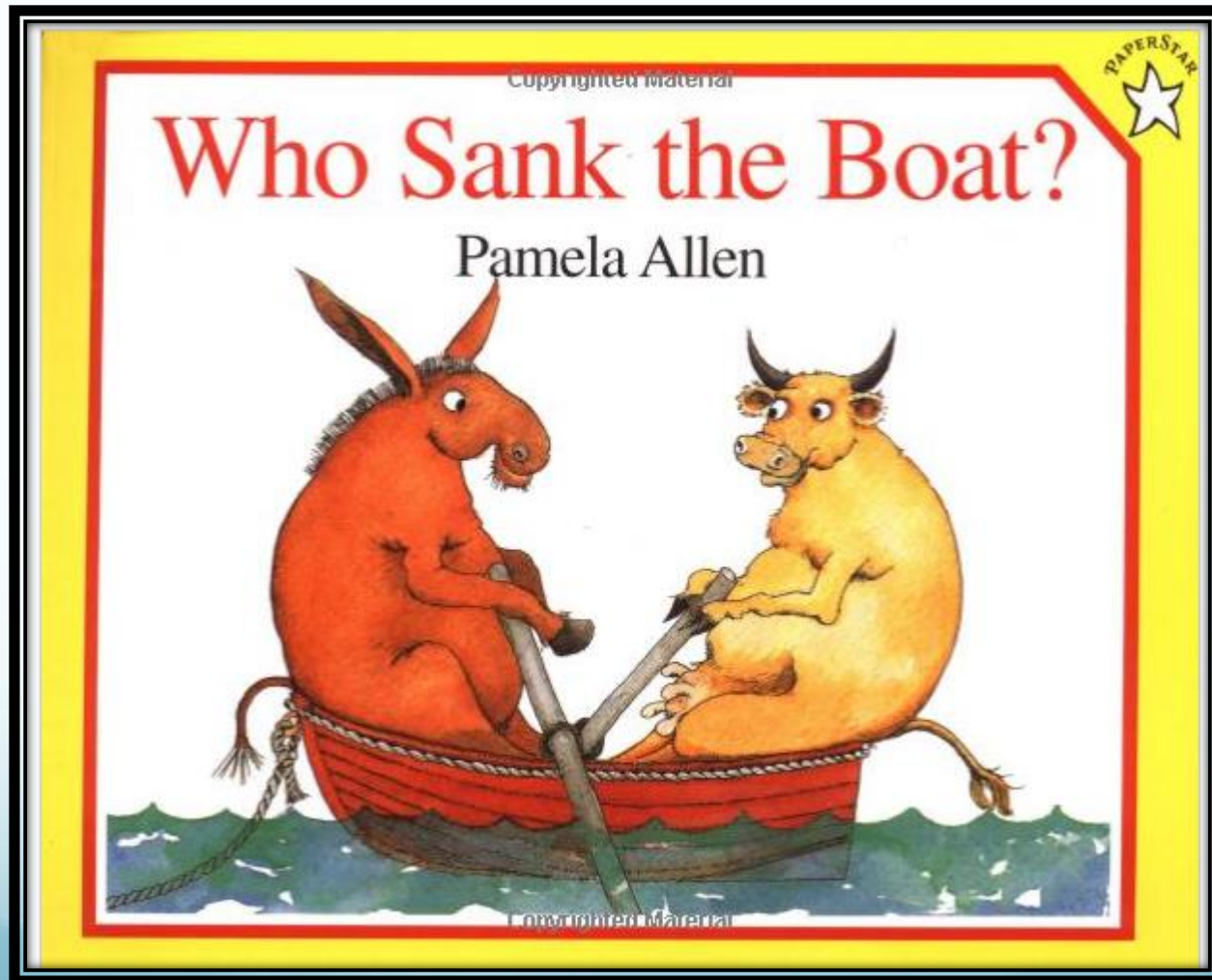
National Library of Virtual Manipulative

<http://nlvm.usu.edu/en/nav/vlibrary.html>

Literature Connections

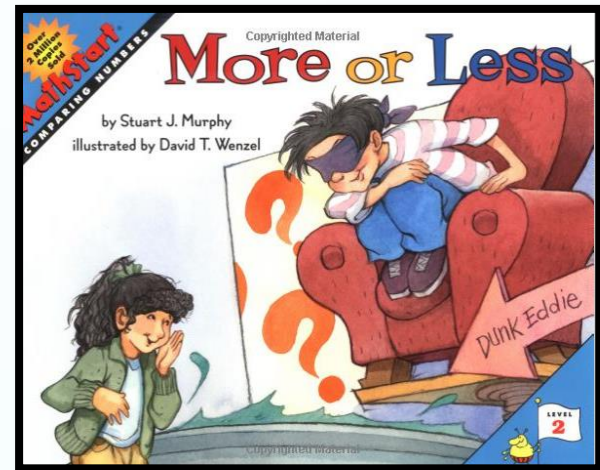
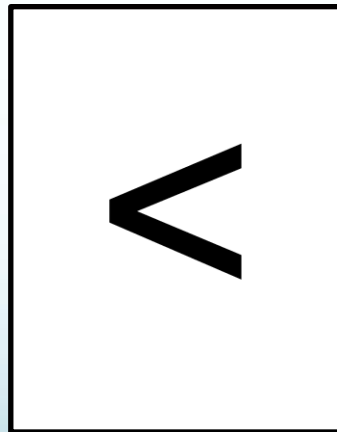


Literature Connections



Literature Connections

Hundreds chart to highlight possible clues leading to the correct age for each person



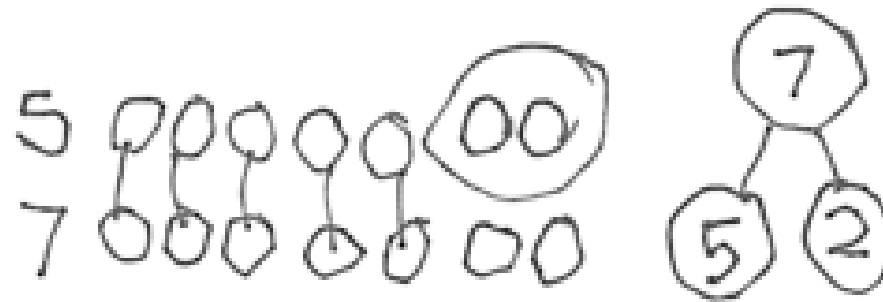


Stations

- Balance Challenges (pan balanced scales station)
 - Explore and create number sentences
 - Balancing Blocks Station (items of varying weights)
- Balance Puzzles
- Balancing Blocks (Pan Balance Exploration)
 - 2 Styrafoam Cubes = 1 Bear, 5 Bears = 1 Glue stick
 - $2 c = 1 b$ $5 b = 1 g$
- Number Balance Exploration
- Writing Equations Swimming Pool Fun



Session Evaluation



$$5 + \textcircled{2} = 7$$

$$7 - 5 = \textcircled{2}$$

He got 2 crackers.

Resources

- Equality. (2003). In *Thinking Mathematically: Integrating Arithmetic and Algebra in Elementary Schools* (pp. 9-24). Portsmouth, NH: Heinemann.
- Falkner, K. P., Levi, L., & Carpenter, T. P. (1999). *Children's Understanding of Equality: A Foundation for Algebra*. *Teaching Children Mathematics*, (1999), 232-236.
- *Get It Together: , Equals, Math Problems for Groups Grades 4-12*. (n.d.). Berkeley, CA: Lawrence Hall of Science.
- Findell, C. (1999). *Groundworks: Algebraic thinking* (pp. 48-63). Chicago, IL: Creative Publications.
- Kroll, V. L., & O'Neill, P. (2005). *Equal, Shmequal*. Watertown, MA: Charlesbridge.
- Murphy, S. J., & Wenzel, D. (2005). *More or less*. New York: HarperCollinsPublishers.
- Satariano, P. (1994). *Storytime mathtime: Math explorations in children's literature*. Palo Alto, CA: Dale Seymour Publications.
- *Who Sank the Boat?* (1982). New York, NY: Sandcastle Books.

Websites

- CaCCSS-M (CA Math Project)
<http://caccssm.cmpso.org>
- K-5 Teaching Resources
<http://www.k-5mathteachingresources.com>
- Inside Mathematics
<http://www.insidemathematics.org>
- National Library of Virtual Manipulatives – Algebra Balance Scales
http://nlvm.usu.edu/en/nav/frames_asid_201_g_4_t_2.html?open=instructions&from=topic_t_2.html
- Riverside County Office of Education
www.rcoe.us/educational-services/instructional-services/california-common-core-state-standards/
- Tales of Frogs and Cupcakes: Think About It! The Equals Sign
<http://frogsandcupcakes.blogspot.com/2011/12/think-about-it-equals-sign.html>